Exercises for Stochastic Processes

Tutorial exercises:

T1. (a) Show that $\mathcal{L}f := \frac{1}{2}f''$ defined on $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$ is a probability generator.

(b) Show that it generates Brownian motion.

(c) Show that the resolvent associated to one-dimensional Brownian motion is given by

$$U_\alpha f(x) = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} f(y) e^{-\sqrt{2\alpha}|x-y|} \mathrm{d}y.$$

T2. Show that the operator on $\{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$ given by

$$\mathcal{L}f := af' + \frac{b}{2}f'',$$

with $a \in \mathbb{R}$, $b \geq 0$, is a probability generator. What is the corresponding semigroup and process?

T3. A probability measure μ is called stationary for a Feller process on S with semigroup (T_t) if $\int T_t f \mathrm{d}\mu = \int f \mathrm{d}\mu$ for all $f \in C_0(S)$ and $t \geq 0$.

(a) How is this definition related to the usual notion of stationarity?

(b) Show that μ is stationary for a process with generator $\mathcal{L}$ if and only if $\int \mathcal{L}f \mathrm{d}\mu = 0$ for all $f \in \mathcal{D}(\mathcal{L})$.

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Homework exercises:

H1. Show that there cannot be two probability generators $\mathcal{L}_1$, $\mathcal{L}_2$ on the same state space with $\mathcal{D}(\mathcal{L}_1) \subsetneq \mathcal{D}(\mathcal{L}_2)$ and $\mathcal{L}_1 = \mathcal{L}_2$ on $\mathcal{D}(\mathcal{L}_1)$.

H2. Let (X_t) be the Feller process with generator $\mathcal{L}f := \frac{1}{2}f'' - f'$ defined on $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f', f'' \in C_0(\mathbb{R})\}$ and start in $x \in \mathbb{R}$. Compute the mean value of its first hitting time of the origin.

H3. Let μT_t denote the distribution at time t of the Feller process with semigroup (T_t) and initial distribution μ. Show that, if $\mu T_t \Rightarrow \nu$ weakly as $t \to \infty$, then ν is stationary.

H4. Let $\varepsilon > 0$ small and let $\mathcal{L}$ be a probability generator. Define for all $f \in C_0(S)$

$$\mathcal{L}_\varepsilon = \mathcal{L}(I - \varepsilon\mathcal{L})^{-1}.$$

(a) Show that $\mathcal{L}_\varepsilon$ is a probability generator.

(b) Show that $\{T_{\varepsilon,t} := \exp(t\mathcal{L}_\varepsilon) \,:\, t \geq 0\}$ is a probability semigroup and

$$\mathcal{L}_\varepsilon = \lim_{t \downarrow 0} \frac{T_{\varepsilon,t} f - f}{t}.$$

Deadline: Tuesday, 07.01.20, 10:00